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# A combined approach of EFGM and precise algorithm in time domain solving viscoelasticity problems

Yang Haitian *, Liu Yan

*State Key Lab of Structural Analysis of Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China*

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**Abstract**

In this paper, element free Galerkin (EFG) method is combined with a precise algorithm in the time domain for solving viscoelasticity problems. By expanding variables at discretized time intervals, an initial boundary value problem is converted into a series of recurrent boundary value problems which can be conveniently solved by EFG/EFG-FE with a self-adaptive computing process. There is no requirement of iteration for the solution of non-linear cases. Satisfactory numerical results are obtained for both static and dynamical viscoelasticity problems.


*Keywords:* Precise algorithm in time domain; EFG method; Viscoelasticity

## 1. Introduction

Viscoelasticity is related with many engineering aspects. The activities in this field have been primarily due to the large scale development and utilization of polymeric materials (see e.g. Christensen, 1982).

The solution of viscoelasticity is time dependent. Due to the complexity of constitutive relationship, boundary condition and boundary geometry etc. although discretized in the space, numerical techniques are still required in the time domain for most cases.

For a specific discretized algorithm in the time domain, there are some decisive factors to be taken into account, such as consistency, convergence and stability. In addition to the above factors, the computing accuracy has to be considered. The computing accuracy is considerably affected by the size of time step that usually may hardly be predicted in many cases.

Yang (1999) presented a new precise algorithm in the time domain, which is self-adaptive for the change of the size of time step, and needs no iteration for the non-linear cases.

It is of interest to combine the above algorithm with the MESHLESS method, which has gained rapid development in recent years, and takes some advantages in solving boundary value problems (see e.g. Belytschko et al., 1994, 1996).

* Corresponding author. Tel./fax: +86-411-4708393.
*E-mail address:* haitian@dlut.edu.cn (Y. Haitian).

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In the field of numerical solution of viscoelasticity, a new attempt is made in this paper by combining the precise algorithm in the time domain with element free Galerkin method (EFGM) and a coupled element free Galerkin (EFG)-FE method. Numerical validations, including both static and dynamical cases, are presented with satisfactory results.

## 2. Recurrent governing equations

The governing equation of dynamical viscoelasticity problems can be described by (see e.g. Christensen, 1982)

$$\sigma_{ij,j} + B_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad x \in \Omega \tag{1}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

where $\sigma_{ij}(\mathbf{x}, t)$ and $\varepsilon_{ij}(\mathbf{x}, t)$ represent tensors of stress and strain, respectively, $u_i(\mathbf{x}, t)$ is the vector of displacement, $\rho(\mathbf{x})$ denotes the mass density, $B_i(\mathbf{x}, t)$ refers to body force, and $\Omega$ is the domain of the problem.

Summation convention is applied to subscribe $j$. For 2D problems, $i$, $j$ range from 1 to 2, for 3D problems $i$, $j$ range from 1 to 3.

The boundary is specified by

$$u_i = \tilde{u}_i \qquad x \in \Gamma_u \tag{3}$$

$$\sigma_{ij} n_j = p_i = \tilde{p}_i \qquad x \in \Gamma_\sigma \tag{4}$$

where $n_j$ represents the unit outside normal, $p_i$ refers to the vector of traction on the boundary, $\tilde{u}_i$, $\tilde{p}_i$ are prescribed functions of $u_i$ and $p_i$, $\Gamma = \Gamma_u + \Gamma_\sigma$ denotes the boundary of $\Omega$, and subscripts $u$ and $\sigma$ denote stress and displacement, respectively.

Initial condition is

$$u_i = \tilde{u}_i^0 \qquad \text{at } t = 0 \tag{5}$$

$$\frac{\partial u_i}{\partial t} = v_i = \tilde{v}_i^0 \qquad \text{at } t = 0 \tag{6}$$

where $\tilde{u}_i^0(\mathbf{x})$ and $\tilde{v}_i^0(\mathbf{x})$ are prescribed functions.

The constitutive relationship of viscoelasticity is described by (see e.g. Yang, 2000)

$$\sigma_{ij} = F(\varepsilon_{lk}) \tag{7}$$

where $F(\varepsilon_{lk})$ is a prescribed function.

Within the time interval $t_0 \leqslant t \leqslant t_0 + T_s$, expanding all variables with term $s$ then yields

$$\sigma_{ij} = \sum_{m=0} \sigma_{ij}^m s^m \tag{8}$$

$$\varepsilon_{ij} = \sum_{m=0} \varepsilon_{ij}^m s^m \tag{9}$$

$$B_i = \sum_{m=0} B_i^m s^m \tag{10}$$

$$u_i = \sum_{m=0} u_i^m s^m \tag{11}$$

$$\tilde{u}_i = \sum_{m=0} \tilde{u}_i^m s^m \tag{12}$$

$$p_i = \sum_{m=0} p_i^m s^m \tag{13}$$

$$\tilde{p}_i = \sum_{m=0} \tilde{p}_i^m s^m \tag{14}$$

where $\sigma_{ij}^m$, $\varepsilon_{ij}^m$, $B_i^m$, $u_i^m$, $\tilde{u}_i^m$, $p_i^m$ and $\tilde{p}_i^m$ represent expanding coefficients of $\sigma_{ij}$, $\varepsilon_{ij}$, $B_i$, $u_i$, $\tilde{u}_i$, $p_i$ and $\tilde{p}_i$, respectively.

$$s = \frac{t - t_0}{T_s} \tag{15}$$

where $t_0$ and $T_s$ refer to the initial point and size of the time interval, respectively.

By virtue of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{T_s}\frac{\mathrm{d}}{\mathrm{d}s} \tag{16}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{1}{T_s^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2} \tag{17}$$

the first and second order derivatives of $T$ with respect to $t$ can be written as

$$\frac{\partial u_i}{\partial t} = \sum_{m=0} \frac{m+1}{T_s} u_i^{m+1} s^m \tag{18}$$

$$\frac{\partial^2 u_i}{\partial t^2} = \sum_{m=0} \frac{(m+2)(m+1)}{T_s^2} u_i^{m+2} s^m \tag{19}$$

Substitution of Eqs. (8)–(19) for Eqs. (1)–(4) then yields

$$\sigma_{ij,j}^m + B_i^m = \frac{(m+1)(m+2)}{T_s^2}\rho u_i^{m+2} \qquad x \in \Omega \tag{20}$$

$$\varepsilon_{ij}^m = \tfrac{1}{2}(u_{i,j}^m + u_{j,i}^m) \tag{21}$$

$$u_i^{m+2} = \tilde{u}_i^{m+2} \qquad x \in \Gamma_u \tag{22}$$

$$\sigma_{ij}^m n_j = p_i^m = \tilde{p}_i^m \qquad x \in \Gamma_\sigma \tag{23}$$

For a kind of viscoelastic materials, Eq. (14) can be written as (see e.g. Yang, 2000)

$$\sigma_{ij}^m = D_{ijkl} \sum_{n=0}^{m} c^{m,m-n} \varepsilon_{kl}^{m-n} \tag{24}$$

where coefficient $c^{m,m-n}$ and tensor $D_{ijkl}$ are constants.

## 3. Implementation of EFGM

By utilizing a weighing residual technique, Eqs. (20)–(24) can be written in a weak form (see e.g. Zienkiewicz and Morgan, 1983).

$$\int_{\Omega}\left(\sigma_{ij,j}^{m}+B_{i}^{m}-\rho\frac{(m+1)(m+2)}{T_{s}^{2}}u_{i}^{m+2}\right)u_{i}^{*}\,\mathrm{d}v+\int_{\Gamma_{u}}(u_{i}^{m+2}-\tilde{u}_{i}^{m+2})p_{i}^{*}\,\mathrm{d}\Gamma-\int_{\Gamma_{\sigma}}(p_{i}^{m}-\tilde{p}_{i}^{m})u_{i}^{*}\,\mathrm{d}\Gamma=0 \tag{25}$$

where $u_i^*$ and $p_i^*$ are weighting functions.

The application of integral by parts for the above equation leads to

$$\frac{(m+1)(m+2)}{T_{s}^{2}}\int_{\Omega}u_{i}^{m+2}\rho u_{i}^{*}\,\mathrm{d}v=\int_{\Omega}B_{i}^{m}u_{i}^{*}\,\mathrm{d}v+\int_{\Gamma_{\sigma}}\tilde{p}_{i}^{m}u_{i}^{*}\,\mathrm{d}\Gamma+\int_{\Gamma_{u}}p_{i}^{m}u_{i}^{*}\,\mathrm{d}\Gamma+\int_{\Gamma_{u}}(u_{i}^{m+2}-\tilde{u}_{i}^{m+2})p_{i}^{*}\,\mathrm{d}\Gamma-\int_{\Omega}\sigma_{ij}^{m}u_{i,j}^{*}\,\mathrm{d}v \tag{26}$$

By using EFG technique, $u_i^m$ and $u_i^*$ are approximated by

$$u_{i}^{m}=[\Phi]\{\bar{u}_{i}^{m}\} \tag{27}$$

$$u_{i}^{*}=[\Phi]\{\bar{u}_{i}^{*}\} \tag{28}$$

where $\{\bar{u}_i^m\}$ and $\{\bar{u}_i^*\}$ are the nodal vectors of $u_i^m$ and $u_i^*$, respectively, and $[\Phi]$ represents a matrix of shape functions.

General nodal vectors are defined by

$$\{\bar{u}^{m}\}^{\mathrm{T}}=\left(\{\bar{u}_{1}^{m}\}^{\mathrm{T}}\quad\{\bar{u}_{2}^{m}\}^{\mathrm{T}}\quad\{\bar{u}_{3}^{m}\}^{\mathrm{T}}\right) \tag{29}$$

$$\{\bar{u}^{*}\}^{\mathrm{T}}=\left(\{\bar{u}_{1}^{*}\}^{\mathrm{T}}\quad\{\bar{u}_{2}^{*}\}^{\mathrm{T}}\quad\{\bar{u}_{3}^{*}\}^{\mathrm{T}}\right) \tag{30}$$

$p_i^m$ and $p_i^*$ can be interpolated along the boundary, having the form

$$p_{i}^{m}=[N]\{\bar{p}_{i}^{m}\} \tag{31}$$

$$p_{i}^{*}=[N]\{\bar{p}_{i}^{*}\} \tag{32}$$

where $[N]$ is a matrix of Lagrangian interpolating functions, $\{\bar{p}_i^m\}$ and $\{\bar{p}_i^*\}$ are the nodal vectors of $p_i^m$ and $p_i^*$, respectively.

The general nodal vectors of $\{\bar{p}_i^m\}$ and $\{\bar{p}_i^*\}$ are defined by

$$\{\bar{p}^{m}\}^{\mathrm{T}}=\left(\{\bar{p}_{1}^{m}\}^{\mathrm{T}}\quad\{\bar{p}_{2}^{m}\}^{\mathrm{T}}\quad\{\bar{p}_{3}^{m}\}^{\mathrm{T}}\right) \tag{33}$$

$$\{\bar{p}^{*}\}^{\mathrm{T}}=\left(\{\bar{p}_{1}^{*}\}^{\mathrm{T}}\quad\{\bar{p}_{2}^{*}\}^{\mathrm{T}}\quad\{\bar{p}_{3}^{*}\}^{\mathrm{T}}\right) \tag{34}$$

Substituting Eqs. (29)–(34) for Eq. (26) with arbitrary $\{\bar{u}^*\}^{\mathrm{T}}$ and $\{\bar{p}^*\}^{\mathrm{T}}$ then yields

$$\begin{bmatrix}A_{m} & B_{m}\\ B_{m}^{\mathrm{T}} & 0\end{bmatrix}\begin{Bmatrix}\bar{u}^{m+2}\\ \bar{p}^{m}\end{Bmatrix}=\begin{Bmatrix}F_{m}\\ f_{m}\end{Bmatrix} \tag{35}$$

where

$$A_{m}=\frac{(m+1)(m+2)}{T_{s}^{2}}\int_{\Omega}[\Phi]^{\mathrm{T}}\rho[\Phi]\,\mathrm{d}v \tag{36}$$

$$F_m = \int_\Omega [\Phi]^{\mathrm{T}}\{B^m\}\,\mathrm{d}v + \int_{\Gamma_2} [\Phi]^{\mathrm{T}}\{\tilde{p}^m\}\,\mathrm{d}v - \int_\Omega [\Phi']^{\mathrm{T}}\{\sigma^m\}\,\mathrm{d}v \tag{37}$$

$$B_m = -\int_{\Gamma_u} [\Phi]^{\mathrm{T}}[N]\,\mathrm{d}\Gamma \tag{38}$$

$$f_m = -\int_{\Gamma_u} [N]^{\mathrm{T}}\{\tilde{u}^{m+2}\}\,\mathrm{d}\Gamma \tag{39}$$

where $[\Phi']$ represents a matrix of derivatives of $[\Phi]$.

By using Eq. (24), Eq. (37) can be expressed as

$$F_m = \int_\Omega [\Phi]^{\mathrm{T}}\{B^m\}\,\mathrm{d}v + \int_{\Gamma_2} [\Phi]^{\mathrm{T}}\{\tilde{p}^m\}\,\mathrm{d}v - \sum_{n=0}^{m}\left(\int_\Omega [\Phi']^{\mathrm{T}}[C^{m,m-n}][\Phi']\,\mathrm{d}v\right)\{\bar{u}^{m-n}\} \tag{40}$$

where matrix $[C^{m,m-n}]$ is associated with $c_{m,m-n}$ and $D_{ijkl}$.

For the case of static viscoelasticity problems, all the terms relevant to the dynamical effect need to be removed.

## 4. Coupled FE-EFG technique

Coupled EF-EFG schemes can be used to avoid inconvenience caused by dealing with essential boundary conditions via EFGM. Belytschko (see e.g. Belytschko, 1991, Belytschko et al., 1995) proposed a variational principle based coupling scheme which is rigorous and accurate, and can pass the patch test. In this paper, to simply illustrate the implementation combining coupled FE-EFG method with the precise algorithm, a coupling technique, which has been developed for the coupling boundary and finite element method (see e.g. Brebbia, 1984, Brebbia et al., 1984, Takao and Hidehero, 1988, Cox, 1988) is employed without constructing the interface element with hybrid displacement approximation in contrast to the scheme (see e.g. Belytschko et al., 1995).

Divide the whole domain into two parts, i.e. EFG domain, and FE domain where all the essential boundaries are assumed to be included.

In FE domain, a recurrent formula can be written as (Yang, 1999)

$$\frac{(m+1)(m+2)}{T_s^2}[M^{\mathrm{FE}}]\{\bar{u}_{\mathrm{FE}}^{m+2}\} = \{F_m^{\mathrm{FE}}\} + \{f_m^{\mathrm{FE}}\} - \sum_{n=0}^{m}[K_{m,m-n}^{\mathrm{FE}}]\{\bar{u}_{\mathrm{FE}}^{m-n}\} + [G]\{\bar{p}_{\mathrm{I}}^m\} \tag{41}$$

where $\{\bar{u}_{\mathrm{FE}}^m\}$ is a nodal vector of expanding coefficient of displacement, $\{\bar{p}_{\mathrm{I}}^m\}$, unknown, refers to a nodal vector of expanding coefficient of traction on the interface.

$$[M^{\mathrm{FE}}] = \sum_e \int_{\Omega_e} [N]^{\mathrm{T}}\rho[N]\,\mathrm{d}v \tag{42}$$

$$\{F_m^{\mathrm{FE}}\} = \sum_e \int_{\Omega_e} [N]^{\mathrm{T}}\{B^m\}\,\mathrm{d}v \tag{43}$$

$$\{f_m^{\mathrm{FE}}\} = \sum_e \int_{\Gamma_\sigma^e} [N]^{\mathrm{T}}\{\tilde{p}^m\}\,\mathrm{d}\Gamma \tag{44}$$

$$[K^{\mathrm{FE}}_{m,m-n}] = \sum_e \sum_{n=0}^{m} \int_{\Omega_e} [N']^{\mathrm{T}} [C^{m,m-n}][N'] \,\mathrm{d}v \tag{45}$$

$$[G] = \sum_e \int_{\Gamma_{\mathrm{I}}^e} [N]^{\mathrm{T}} [N] \,\mathrm{d}\Gamma \tag{46}$$

$[N]$ and $[N']$ denote matrixes of shape functions and their derivatives, respectively, summation $\sum_e$ covers all finite elements.

Re-write Eq. (41) in the form

$$[A^{\mathrm{FE}}_m]\{\bar{u}^{m+2}_{\mathrm{FE}}\} = \{b^{\mathrm{FE}}_m\} + [G]\{\bar{p}^m_{\mathrm{I}}\} \tag{47}$$

where

$$[A^{\mathrm{FE}}_m] = \frac{(m+1)(m+2)}{T_s^2}[M^{\mathrm{FE}}] \tag{48}$$

$$\{b^{\mathrm{FE}}_m\} = \{F^{\mathrm{FE}}_m\} + \{f^{\mathrm{FE}}_m\} - \sum_{n=0}^{m} [K^{\mathrm{FE}}_{m,m-n}]\{u^{m-n}_{\mathrm{FE}}\} \tag{49}$$

By making use of

$$(p^m_{\mathrm{I}})^{\mathrm{EFG}} = -(p^m_{\mathrm{I}})^{\mathrm{FE}} = -[N]\{(\bar{p}^m_{\mathrm{I}})\}$$

Eq. (32) in the EFG domain can be rewritten as

$$\frac{(m+1)(m+2)}{T_s^2}[M^{\mathrm{EFG}}]\{\bar{u}^{m+2}_{\mathrm{EFG}}\} = \{F^{\mathrm{EFG}}_m\} + \{f^{\mathrm{EFG}}_m\} - \sum_{n=0}^{m} [K^{\mathrm{EFG}}_{m,m-n}]\{\bar{u}^{m-n}_{\mathrm{EFG}}\} - [J_{\mathrm{EFG}}]\{\bar{p}^m_{\mathrm{I}}\} \tag{50}$$

where $\{\bar{u}^m_{\mathrm{EFG}}\}$ denotes the nodal vector of expanding coefficient of displacement, $\{p^m_{\mathrm{I}}\}$ denotes the vector of traction on the interface, both subscripts and superscripts FE and EFG refer to EF domain and EFG domain, respectively.

$$[M^{\mathrm{EFG}}] = \int_{\Omega_{\mathrm{EFG}}} [\Phi]^{\mathrm{T}} \rho [\Phi] \,\mathrm{d}v \tag{51}$$

$$\{F^{\mathrm{EFG}}_m\} = \int_{\Omega_{\mathrm{EFG}}} [\Phi]^{\mathrm{T}} \{B_m\} \,\mathrm{d}v \tag{52}$$

$$\{f^{\mathrm{EFG}}_m\} = \int_{\Gamma_\sigma^{\mathrm{EFG}}} [\Phi]^{\mathrm{T}} \{\tilde{p}_m\} \,\mathrm{d}\Gamma \tag{53}$$

$$[K^{\mathrm{EFG}}_{m,m-n}] = \int_{\Omega_{\mathrm{EFG}}} [\Phi']^{\mathrm{T}} [C_{m,m-n}][\Phi'] \,\mathrm{d}v \tag{54}$$

$$[J_{\mathrm{EFG}}] = \sum_e \int_{\Gamma_{\mathrm{I}}^e} [\Phi]^{\mathrm{T}} [N] \,\mathrm{d}\Gamma \tag{55}$$

Eq. (50) can further be written as

$$[A^{\mathrm{EFG}}_m]\{\bar{u}^{m+2}_{\mathrm{EFG}}\} = \{b^{\mathrm{EFG}}_m\} - [J_{\mathrm{EFG}}]\{\bar{p}^m_{\mathrm{I}}\} \tag{56}$$

where

$$[A_m^{\mathrm{EFG}}] = \frac{(m+1)(m+2)}{T_s^2}[M^{\mathrm{EFG}}] \tag{57}$$

$$\{b_m^{\mathrm{EFG}}\} = \{F_m^{\mathrm{EFG}}\} + \{f_m^{\mathrm{EFG}}\} - \sum_{n=0}^{m}[K_{m,m-n}^{\mathrm{EFG}}]\{\bar{u}_{\mathrm{EFG}}^{m-n}\} \tag{58}$$

In the FE domain, separating the nodes at the interface from others then yields

$$\{\bar{u}_{\mathrm{FE}}^m\}^{\mathrm{T}} = \left(\{\bar{u}_{\mathrm{FE}}^m\}_a^{\mathrm{T}} \quad \{\bar{u}_{\mathrm{FE}}^m\}_b^{\mathrm{T}}\right) \tag{59}$$

where subscripts $b$ and $a$ refer to the nodes at the interface and others else, respectively.

Since $[G]$ is only available at the interface Eq. (47) can be written in the form

$$\begin{bmatrix} (A_m^{\mathrm{FE}})_{aa} & (A_m^{\mathrm{FE}})_{ab} \\ (A_m^{\mathrm{FE}})_{ba} & (A_m^{\mathrm{FE}})_{bb} \end{bmatrix} \begin{Bmatrix} (\bar{u}_{\mathrm{FE}}^{m+2})_a \\ (\bar{u}_{\mathrm{FE}}^{m+2})_b \end{Bmatrix} = \begin{Bmatrix} (b_m^{\mathrm{FE}})_a \\ (b_m^{\mathrm{FE}})_b \end{Bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} \{\bar{p}_{\mathrm{I}}^m\} \tag{60}$$

$\{\bar{p}_{\mathrm{I}}^m\}$ can be obtained via Eq. (60), having the form

$$\{\bar{p}_{\mathrm{I}}^m\} = [G]^{-1}\left([(A_m^{\mathrm{FE}})_{ba}]\{(\bar{u}_{\mathrm{FE}}^{m+2})_a\} + [(A_m^{\mathrm{FE}})_{bb}]\{(\bar{u}_{\mathrm{FE}}^{m+2})_b\} - \{(b_m^{\mathrm{FE}})_b\}\right) \tag{61}$$

Substituting Eq. (61) for Eq. (56) with a rearrangement can yield

$$[A_m]\{\bar{u}^{m+2}\} = \{b_m\} \tag{62}$$

where

$$[A_m] = \begin{bmatrix} A_m^{\mathrm{EFG}} + J_{\mathrm{EFG}}J_{\mathrm{FE}}^{-1}(A_m^{\mathrm{FE}})_{bb}H & J_{\mathrm{EFG}}J_{\mathrm{FE}}^{-1}(A_m^{\mathrm{FE}})_{ba} \\ (A_m^{\mathrm{FE}})_{ab}H & (A_m^{\mathrm{FE}})_{aa} \end{bmatrix} \tag{63}$$

$$\{b_m\} = \begin{Bmatrix} b_m^{\mathrm{EFG}} + J_{\mathrm{EFG}}J_{\mathrm{FE}}^{-1}(b_m^{\mathrm{FE}})_b \\ (b_m^{\mathrm{FE}})_a \end{Bmatrix} \tag{64}$$

$$\{\bar{u}^{m+2}\}^{\mathrm{T}} = \left(\{\bar{u}_{\mathrm{EFG}}^{m+2}\}^{\mathrm{T}} \quad \{(\bar{u}_{\mathrm{FE}}^{m+2})_a\}^{\mathrm{T}}\right) \tag{65}$$

$$\{(\bar{u}_{\mathrm{FE}}^m)_b\} = [H]\{\bar{u}_{\mathrm{EFG}}^m\} \tag{66}$$

$$[H]^{\mathrm{T}} = \left[\,[\Phi]|_{\mathrm{node}\,1} \quad [\Phi]|_{\mathrm{node}\,2} \quad \cdots \quad [\Phi]|_{\mathrm{node}\,n_{\mathrm{I}}}\,\right]^{\mathrm{T}} \tag{67}$$

$n_{\mathrm{I}}$ is an integer representing the number of nodes at the interface.

## 5. Numerical validation

**Example 1** (*Free vibration of a viscoelastic rod*). Consider a free vibration of a viscoelastic rod with $\rho = 100.0$ and $L = 0.9$.

A Kelvin model with $q_0 = 1.0$ and $q_1 = 2.0$, is used in the computation.

The initial condition is

$$u(x,0) = \tilde{u}^0(x) = 0$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v(x,0) = \tilde{v}^0(x) = 0.75\frac{x}{L}$$

In the computing

$$T_s = 0.2 \text{ s}, \qquad \beta = 1 \times 10^{-4}$$

15 equidistant nodes and two finite elements are employed.

Numerical comparisons with a FE based precise algorithm (see e.g. Gao, 1999) are exhibited from Figs. 2–6.

**Example 2** (1*D static viscoelastic problems*). A viscoelastic rod, as shown in Fig. 1, is subjected to a load $P(t)$ at the right end. Various solutions of displacement are obtained for different constitutive models, and compared with analytical solutions (see e.g. Gao, 1999).

In the computing $\beta = 0.0001$.

The initial condition is

$$u(x,0) = \tilde{u}^0(x) = 0$$

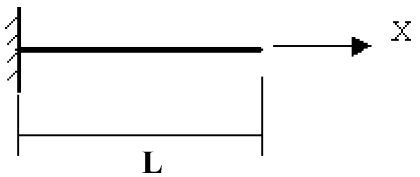


Fig. 1. A viscoelastic rod.

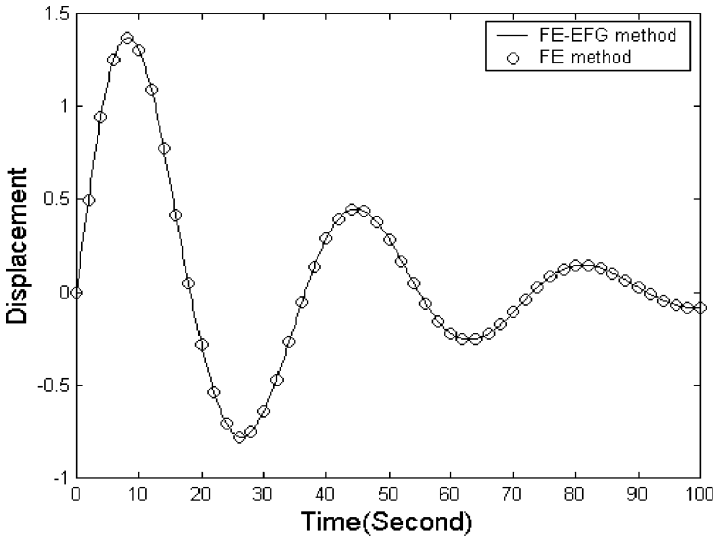


Fig. 2. Numerical comparison of displacement at $x = 0.3$.

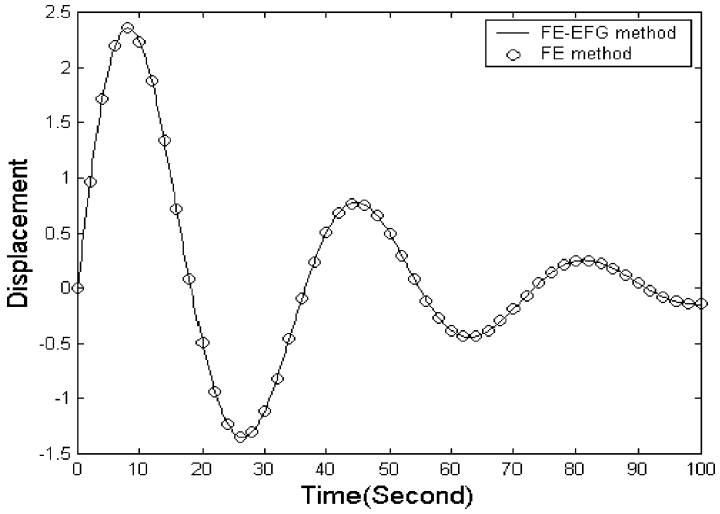


Fig. 3. Numerical comparison of displacement at $x = 0.6$.

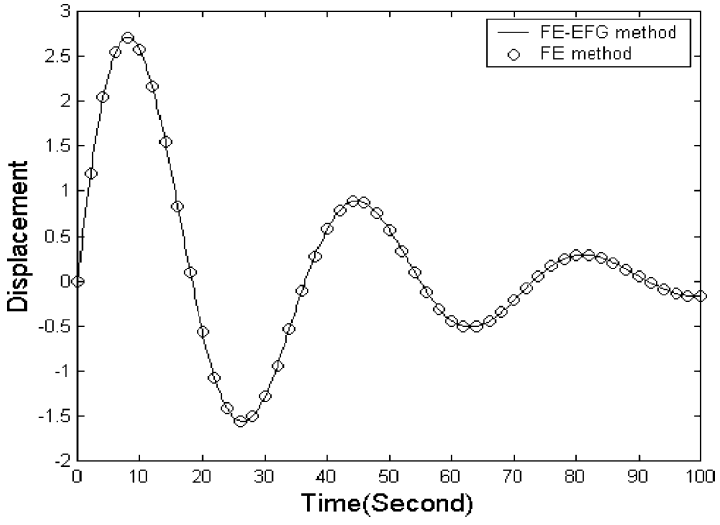


Fig. 4. Numerical comparison of displacement at $x = 0.9$.

(1) *Linear model:*

$$P(t) = \begin{cases} 1.0 & 0 \leqslant t \leqslant 4.0 \\ 0 & t > 4.0 \end{cases}$$

Computing parameters are given in Table 1, numerical comparisons are exhibited in Table 2 and Fig. 7.

(2) *Linear model:*

$$P(t) = P_0 H(t)$$

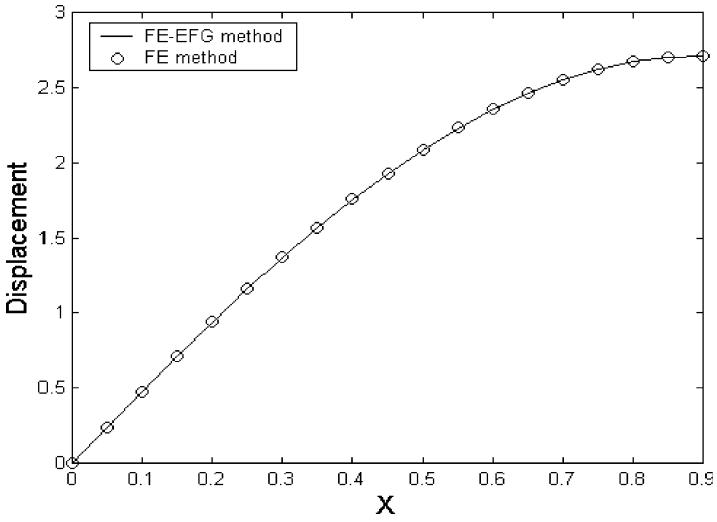


Fig. 5. Numerical comparison of displacement at $t = 8.2$ s.

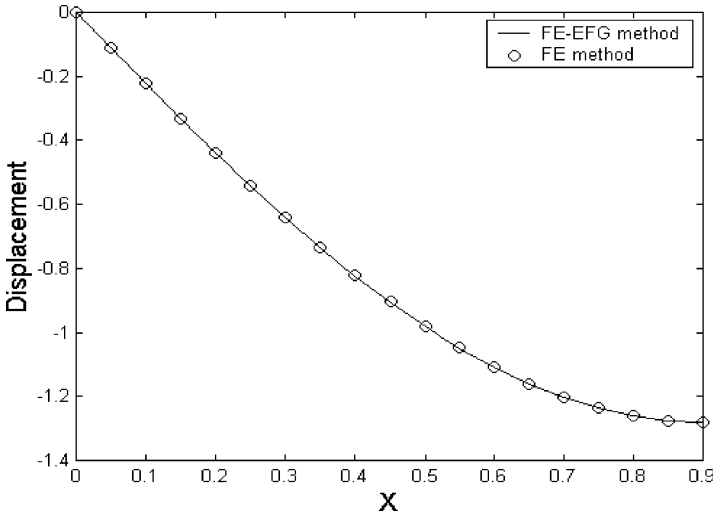


Fig. 6. Numerical comparison of displacement at $t = 30.0$ s.

Table 1
Computing parameters

| Parameters | $p_1$ | $q_0$ | $q_1$ | $L$ | $T_s$ | Number of nodes |
|---|---|---|---|---|---|---|
| Values | 0.5 | $4 \times 10^5$ | $8 \times 10^5$ | 1.0 | 0.1 | 11 |

Table 2
Numerical comparison of displacement at $t = 3.0$, 6.0

| $x$ | $t = 3.0$ | | $t = 6.0$ | |
|---|---|---|---|---|
| | Presented method (EFG) | Analytical solution | Presented method (EFG) | Analytical solution |
| 0.2 | 4.1631E−07 | 4.1633E−07 | 1.1928E−007 | 1.1928E−007 |
| 0.5 | 1.0408E−06 | 1.0408E−06 | 2.9821E−007 | 2.9821E−007 |
| 0.7 | 1.4571E−06 | 1.4571E−06 | 4.1749E−007 | 4.1750E−007 |
| 0.9 | 1.8734E−06 | 1.8735E−06 | 5.3676E−007 | 5.3678E−007 |
| 1.0 | 2.0816E−06 | 2.0816E−06 | 5.9642E−007 | 5.9642E−007 |

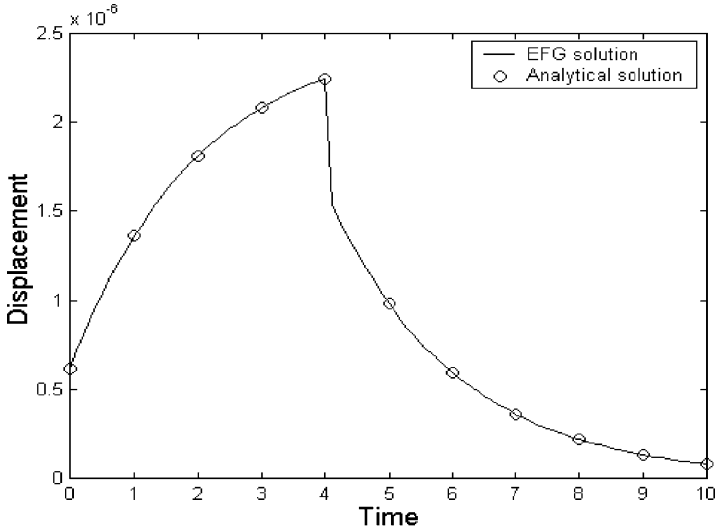


Fig. 7. Numerical comparison of displacement at $x = 0.9$.

Table 3
Computing parameters

| Parameters | $p_1$ | $q_0$ | $q_1$ | $L$ | $P$ | Number of nodes | Number of elements |
|---|---|---|---|---|---|---|---|
| Values | 1.0 | $1.2 \times 10^5$ | $2.0 \times 10^5$ | 1.0 | 1000.0 | 10 | 1 |

where

$$H(t) = 0 \qquad t < 0$$

$$H(t) = 1 \qquad t \geqslant 0$$

Computing parameters are given in Table 3, numerical comparisons are shown in Tables 4 and 5.

(3) *Kelvin model:*

$$P(t) = P_0 H(t)$$

Computing parameters are listed in Table 6, numerical comparisons are exhibited in Tables 7 and 8.

Table 4
Numerical comparison of displacement at $x = 1.0$, 0.7

| $x$ | Size of time step | $t$ | Presented method (EFG-FE) | Analytical solution |
|---|---|---|---|---|
| 1.0 | 0.1 | 0.1 | 5.1941E−003 | 5.1941E−003 |
| | | 0.2 | 5.3769E−003 | 5.3769E−003 |
| | | 0.3 | 5.5491E−003 | 5.5491E−003 |
| | | 0.4 | 5.7112E−003 | 5.7112E−003 |
| | | 0.5 | 5.8639E−003 | 5.8639E−003 |
| | 0.5 | 0.5 | 5.8639E−003 | 5.8639E−003 |
| | | 1.0 | 6.5040E−003 | 6.5040E−003 |
| | | 1.5 | 6.9781E−003 | 6.9781E−003 |
| | | 2.0 | 7.3293E−003 | 7.3294E−003 |
| | | 2.5 | 7.5895E−003 | 7.5896E−003 |
| 0.7 | 0.2 | 0.2 | 3.7638E−003 | 3.7639E−003 |
| | | 0.4 | 3.9979E−003 | 3.9979E−003 |
| | | 0.6 | 4.2054E−003 | 4.2054E−003 |
| | | 0.8 | 4.3895E−003 | 4.3895E−003 |
| | | 1.0 | 4.5528E−003 | 4.5528E−003 |
| | 0.4 | 0.4 | 3.9978E−003 | 3.9979E−003 |
| | | 0.8 | 4.3895E−003 | 4.3895E−003 |
| | | 1.2 | 4.6975E−003 | 4.6976E−003 |
| | | 1.6 | 4.9399E−003 | 4.9399E−003 |

Table 5
Numerical comparison of displacement at $t = 5.0$

| Size of time step | $x$ | Presented method (EFG-FE) | Analytical solution |
|---|---|---|---|
| 5.0 | 0.2 | 1.6338E−003 | 1.6335E−003 |
| | 0.4 | 3.2670E−003 | 3.2670E−003 |
| | 0.6 | 4.9004E−003 | 4.9004E−003 |
| | 0.8 | 6.5339E−003 | 6.5339E−003 |
| | 1.0 | 8.1673E−003 | 8.1674E−003 |

Table 6
Computing parameters

| Parameters | $q_0$ | $q_1$ | $L$ | $P$ | Number of nodes | Number of elements |
|---|---|---|---|---|---|---|
| Values | $5.0 \times 10^5$ | $2.4 \times 10^5$ | 1.0 | 4000.0 | 10 | 1 |

Table 7
Numerical comparison of displacement at $x = 0.5$, 0.8

| $x$ | Size of time step | $t$ | Presented method (EFG-FE) | Analytical solution |
|---|---|---|---|---|
| 0.5 | 0.3 | 0.3 | 1.8590E−003 | 1.8590E−003 |
| | | 0.6 | 2.8540E−003 | 2.8540E−003 |
| | | 0.9 | 3.3866E−003 | 3.3866E−003 |
| | | 1.2 | 3.6716E−003 | 3.6717E−003 |
| | | 1.5 | 3.8243E−003 | 3.8243E−003 |
| 0.8 | 0.9 | 0.9 | 5.4186E−003 | 5.4185E−003 |
| | | 1.8 | 6.2495E−003 | 6.2495E−003 |
| | | 2.7 | 6.3769E−003 | 6.3769E−003 |
| | | 3.6 | 6.3966E−003 | 6.3965E−003 |

Table 8
Numerical comparison of displacement at $t = 3.0$

| Size of time step | $x$ | Presented method (EFG-FE) | Analytical solution |
|---|---|---|---|
| 3.0 | 0.1 | 7.9847E−004 | 7.9846E−004 |
| | 0.3 | 2.3953E−003 | 2.3954E−003 |
| | 0.5 | 3.9923E−003 | 3.9923E−003 |
| | 0.7 | 5.5893E−003 | 5.5892E−003 |
| | 0.9 | 7.1859E−003 | 7.1861E−003 |

## 6. Conclusions

The major objective of this paper is to make a new attempt combining EFGM with a precise algorithm in the time domain to solve viscoelasticity problems, the merits of this combination include

1. An initial boundary value problem is converted into a series of recurrent boundary value problems which are solved by EFGM as in the case of static elasticity. In addition to EFGM, other well developed numerical approaches can also be adopted with a specific consideration of characteristics of the problem, including in-homogeneity, complex boundary shape, and boundary conditions etc.
2. A self-adaptive computation can be realized by using the precise algorithm in the time domain, which results in a more precise description for the variation of variables, and compensates the possible lost computing accuracy caused by improper choices of the size of time step.
3. For non-linear cases, no any assumption is made, and no any iteration is needed.
4. By utilizing a coupling technique developed for the coupling FE-BE method, the implementation combining coupled FE-EFG methods with the precise algorithm in time domain is simply illustrated. Mathematically Eqs. (20)–(24) can also definitely be solved via the coupling approach proposed by Belytschko (see e.g. Belytschko, 1991, Belytschko et al., 1995), which is more accurate than the coupling technique used in this paper (see e.g. Belytschko, 1991), and is being employed by authors for solving 2D viscoelasticity problems with an expectation of more perfect results.

Numerical validation, including both static and dynamical cases, presents a good accordance with other solutions, which do encourage authors to extend present work to other fields. In fact, some preliminary results have been achieved for the solutions of heat conduction problems, and will be presented somewhere else.

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